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Renormalization of nonpolynomial Lagrangians

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Abstract. With Chiral Lagrangians in mind, we study Lagrangians whose Dyson index is non-negative, for which the problem is the removal of the ultraviolet divergencies that remain after the usual resummation over the minor coupling constant. We have shown that for any rational interaction Lagrangian without derivatives and of Dyson index less than four, these divergences may be removed in all orders by local counter-terms. This is a special case of a more general result, namely that the method of local counter-terms works if $\mathscr{L}_{\text{int}} = \mathscr{L}_{\text{p}} + \mathscr{L}_{\text{np}}$, where $\mathscr{L}_{\text{p}}$ is a polynomial of Dyson index less than four and $\mathscr{L}_{\text{np}}$ is a nonpolynomial for which the resummation leaves no divergences. We present the counter-terms explicitly after resummation over the minor coupling constant.

1. Introduction

There has been a great deal of interest recently in the use of nonpolynomial Lagrangians in quantum field theory. This stems from the nonlinear representations of chiral groups, such as $SU(2) \times SU(2)$ or $SU(3) \times SU(3)$—see, for example, Geffen and Gasiorowicz (1969) and references therein—and from the nonpolynomial forms of weak interactions and general relativity (Delbourgo *et al.* unpublished†). Initial calculations for the chiral case have been made only in the tree approximation, and coincide with the current algebra results. Our interest is in what can be said about such nonpolynomial Lagrangians when closed loops are included. In other words we wish to take these nonpolynomial Lagrangians seriously as models of quantum field theory. The main problem facing such an approach is that of ultraviolet divergences, and in particular their removal by some type of renormalization.

The usual perturbation solution of such problems is hopelessly inadequate in the nonpolynomial case, since expansion of the interaction Lagrangian in powers of the field variables will generate highly non-renormalizable interactions; these will then produce arbitrarily high divergences in S-matrix elements. It is certainly possible to regularize such series term by term (Steinmann 1964, 1966), but at the expense of introducing an infinite number of arbitrary constants. Instead of doing this, attempts have recently been made to regularize the sum of the series directly, either by avoiding the initial perturbation expansion of the interaction Lagrangian or by suitable resummation methods (Efimov unpublished, Okubo 1954, Fradkin 1963, Volkov 1969 and references therein, Salam 1970). When the series are divergent, formal sums have been used.

It is not yet certain that the results of such manipulation have physical content. Unitarity has been investigated, and verified for a particular case (Volkov 1968), as has the appropriate analyticity for another (Lee and Zumino 1969). Asymptotic behaviour has also been investigated for scalar interactions (Strathdee and Salam 1970), and the dipole asymptotic behaviour of the nucleon electromagnetic form factor shown to be valid for chiral $SU(2) \times SU(2)$ (Martin and Taylor

† Delbourgo, R., Salam, A., and Strathdee, J., *Infinities in Einstein's Gravitational Theory*, Imperial College preprint ICTP/69/28.

1970). However, the field theories being discussed appear, in general, to be outside the class of localizable fields considered by Jaffe (1966, 1967); this means that the usual properties of analyticity, TCP, spin and statistics, etc., need no longer be valid. However, recent attempts have been made to repair this damage (Steinmann unpublished,† Taylor unpublished‡); in particular one of us (J.G.T.) has shown that rational functions of the free field can be included in a theory of nonlocalizable fields for which the majority of the results for localizable fields can, in fact, be derived. However, for this approach to apply it is necessary that the Wightman functions of the theory be the limits, in a suitable sense, of those arising from localizable fields. It is not at all evident that the approach is valid for the nonpolynomial Lagrangians we consider here; provided they are always constructed from local products of field operators there is a chance that it may.

We will proceed here to discuss the renormalization of these nonpolynomial Lagrangians. This is a necessary part of the program of making sense of these models, and is essential in order to understand their physical predictions. Chiral Lagrangians are typically rational functions of the fields and their derivatives, and for simplicity we will only consider here interactions which are rational functions of the fields. We will also restrict ourselves to neutral spinless isoscalar particles. For this case the simplest rational nonpolynomial Lagrangian is $g(1+\lambda\phi)^{-1}$. Since any rational function of the field ϕ can by means of partial fractions be expressed as a sum of such terms plus a polynomial, we need initially only consider the interaction $P(\phi)+g(1+\lambda\phi)^{-1}$, for some polynomial P. Recent conjectures by Salam and co-workers (Salam 1970) have been made to the effect that such theories are renormalizable by addition of suitable counter terms, if the degree of the polynomial P is less than four, and a conjectured form of these counter terms has been given in a more recent work (Delbourgo *et al.* unpublished§). However, the justification given there is incomplete in several aspects.

We give here a complete discussion of the various problems which arise in this situation, in particular showing how nonlocal divergent terms are correctly cancelled by suitable counter-terms. We extend our discussion to give the counter-terms for the more general case of a rational function for which the numerator has degree at most three more than the denominator, and conclude with a brief discussion of the case when this is no longer true.

2. Resummation methods

We consider first the interaction Lagrangian

$$\mathscr{L}_{\text{int}} = \frac{g}{1+\lambda\phi}. \tag{1}$$

We use the method of Borel summation, that is, of replacing

$$\sum_n a_n n! \qquad \text{by} \qquad \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t} \sum_n a_n t^n$$

† Steinmann, O., 1970, *Scattering formalism for Non-localisable fields*, S.I.N., Zurich preprint.

‡ Taylor, J. G., 1969, *Local Commutativity for non-localisable fields*, University of Southampton preprint.

§ Delbourgo, R., Koller, K., and Salam, A., *Renormalization of Rational Lagrangians*, Imperial College preprint ICTP/69/10.

where free interchange of integration and summation has been effected. This turns out to be equivalent to replacing the interaction Lagrangian (1) by

$$g\int_0^\infty \mathrm{d}t\,\exp\{-t(1+\lambda\phi)\}. \tag{2}$$

The essential point is the appearance of the field in (2) linearly in the exponent. This has been used previously by Efimov (unpublished), but by means of Fourier transforms. This method has the disadvantage that ultraviolet divergences are not automatically damped out, as they will be by the use of the representation (2).

The S-matrix is, using Hori's (1952) formula,

$$S = \exp\left(\tfrac{1}{2}\frac{\delta}{\delta\phi}\Delta\frac{\delta}{\delta\phi}\right)\exp(\mathrm{i}\mathscr{L}(\phi)) \tag{3}$$

$$= \sum_{N\geqslant 0}\frac{(ig)^N}{N!}\int \mathrm{d}\overset{N}{X}\mathrm{Sk}(\overset{N}{X}) \tag{4}$$

where

$$\mathrm{Sk}(\overset{N}{X}) = \int \mathrm{d}\overset{N}{t}\prod_{i=1}^{N}\exp\{-t_i(1+\lambda_i\phi_i)\}\prod_{i<j}\exp(\lambda^2 t_i t_j\Delta_{ij}) \tag{5}$$

and we are using the abbreviations

$$\frac{\delta}{\delta\phi}\Delta\frac{\delta}{\delta\phi} = \int \mathrm{d}x\int \mathrm{d}y\Delta(x-y)\frac{\delta}{\delta\phi(x)}\cdot\frac{\delta}{\delta\phi(y)}$$

$$\mathscr{L}(\phi) = \int \mathrm{d}x\,\mathscr{L}_{\mathrm{int}}(\phi)$$

$$\overset{N}{X} = (x_1,\ldots,x_N),\quad \phi_i = \phi(x_i),\quad \Delta_{ij} = \Delta(x_i-x_j) = \langle\phi(x_i)\phi(x_j)\rangle_{+0}.$$

There is a certain degree of ambiguity in the definition of the skeleton function $\mathrm{Sk}(\overset{N}{X})$ in (4) due to the divergence of the function $\Delta(x)$ as $x\to 0$. This problem may be resolved by means of Fourier transforms, as has been shown by Volkov (1968). We will stay in coordinate space by an alternative approach; we rotate the contours of integration for the variables t_i by a common angle θ. Provided that $\cos\theta > 0$, $\cos 2\theta < 0$, or $\pi/4 < |\theta| < \pi/2$, then we have the exponential factors $\exp(-t_i)$ still produce damping, whilst the singular term in the exponent, $t_i t_j\Delta_{ij}$, has real part $\cos 2\theta\, t_i t_j\Delta_{ij}$, where the t_i are still real. This is negative for the Euclidean region, since there all the Δ_{ij} are positive (independently of the mass m of the meson). Thus the skeleton functions have no ultraviolet divergences. We define the functions in the Lorentz region by analytic continuation, for example by means of their Fourier transforms, following Volkov (1968). There is still an ambiguity, since the rotation can be through either a common positive or negative angle θ; we may add together the two resulting continuations, multiplied by arbitrary constants. Unitarity reduces the resulting arbitrariness a little (Strathdee and Salam unpublished, Volkov 1968); we will not consider that further here.

We now turn to the case of a general rational function of ϕ, which may be written as

$$\mathscr{L}_{\mathrm{int}}(\phi) = P(\phi)+\sum_n\frac{A_n}{(1+\lambda_n\phi)}$$

where $P(\phi)$ is a polynomial in ϕ, and A_n, λ_n are constants. We only consider single powers of $(1+\lambda_n\phi)^{-1}$, since higher powers of this term may be expressed by means of derivatives with respect to λ_n acting on it. We may consider, then, that $\mathscr{L}_{\text{int}}$ is a sum of a polynomial and a nonpolynomial part; we can draw the general Feynman diagram resulting from this separation as in figure 1. This is described precisely by the factorization theorem proved in Appendix 1,

$$S(\phi) = S_{\text{p}}\left(\psi+\Delta\frac{\delta}{\delta\phi}\right)S_{\text{np}}(\phi)|_{\psi=\phi} \tag{6}$$

where $S_{\text{np}}(\phi)$ is derived by using the expression (3) with $\mathscr{L}(\phi)$ replaced by

$$\mathscr{L}_{\text{np}}(\phi) = \int \mathrm{d}^4x \sum_n \frac{A_n}{(1+\lambda_n\phi(x))}. \tag{7}$$

and $S_{\text{p}}(\phi)$ by using $\mathscr{L}_{\text{p}}\ (\phi)$.

We may expand $S_{\text{np}}(\phi)$ in a form similar to the skeletal expansion (4), where now there are mixed terms, with the skeletal vertices x_i in (5) being associated with the

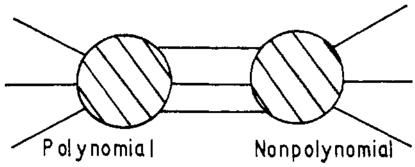

Figure 1. The separation of the polynomial and nonpolynomial vertices in a general Feynman diagram, giving a graphical form of the factorization expression of equation (6).

various terms in the expansion of (7). We thus have to associate with each vertex x_i an integer n_i specifying that term in (7) which is used at this vertex, and so the skeletal contribution becomes

$$Sk(X) = \sum^{N} \int \mathrm{d}t \prod_i^{N} A_{ni} \exp\{-t_i(1+\lambda_i\phi_i)\} \prod_{i<j} \exp(\lambda_{n_i}\lambda_{n_j}t_it_j\Delta_{ij}) \tag{8}$$

where the summation is over all possible choices of integers n_i from the set entering on the right hand side of (7). The rotation of the contours of integration over the variables t_i to obtain a well defined function in the euclidean coordinate region goes as before, though with angles of rotation depending upon which term in the summation is chosen in (8) and on the variable t_i being considered. Specifically the angle of rotation θ_i for the integration over t_i in a given term in the sum (8) must satisfy the condition

$$\pi/2 < |\alpha_i+\alpha_j+\theta_i+\theta_j| < \pi \qquad |\theta_i| < \pi/2 \tag{9}$$

for any pair i, j, where α_i is the phase of λ_{n_i}. This cannot be satisfied in general, but can for the simplest case when the constants λ_n are real, when $\pi/4 < \theta_i < \pi/2$ if λ_{n_i} is positive and $-\pi/4 > \theta_i > -\pi/2$ if λ_{n_i} is negative. There is still the problem, as before, of continuation from the Euclidean to the Lorentz region; we may still achieve this, say, by going to momentum space. This will again ensure correct analyticity, and most likely correct discontinuities across physical cuts, though will produce nonlocalizable fields as in the earlier case (Steinmann unpublished).

This will also allow the nonpolynomial part of the nonlinear σ model to be made convergent. The more general case when the contours of integration cannot be defined rotated to satisfy (9) may still possibly be defined by an extension of Volkov's (1968) method to the many-variable case.

3. The divergences

Let us consider how divergences may arise when we have the interaction Lagrangian $\mathscr{L}_{\text{int}} = \mathscr{L}_{\text{p}} + \mathscr{L}_{\text{np}}$. We will assume that the S matrix arising from $\mathscr{L}_{\text{np}}$ by the methods of the last section contains no ultraviolet divergences after resummation in the minor coupling constants λ_{n}. Our further discussion will, in fact, be independent of the explicit form of $\mathscr{L}_{\text{np}}$, except for certain points which we will specify later. The ultraviolet divergences which we wish to consider arise from the powers of the propagator occurring in $S_{\text{p}}(\psi + \Delta\delta/\delta\phi)$. We can see how this occurs graphically by realizing that the action of $S_{\text{p}}(\psi + \Delta\delta/\delta\phi)$ is obtained by drawing all the perturbation graphs for the polynomial interaction $\mathscr{L}_{\text{p}}$ and interpreting each external line as $(\psi + \Delta\delta/\delta\phi)$. Thus the divergences inherent in $\mathscr{L}_{\text{p}}$ will be present, as well as new ones caused by the action of the powers of $\Delta\delta/\delta\phi$ which are present. The former divergences can only be subtracted with a finite number of counter terms if the degree of the polynomial $P(\phi)$ is no more than four. We will restrict ourselves in the following to the case when this degree is three (the case of degree two is very similar); we will give a brief discussion in § 4 when this degree is four.

We thus consider

$$\mathscr{L}_{\text{int}} = g\phi^3 + \mathscr{L}_{\text{np}}(\phi).$$

To zeroth order in $\mathscr{L}_{\text{np}}$ we have that the S-operator is

$$S_{\text{p}}\left(\psi + \Delta\frac{\delta}{\delta\phi}\right)1|_{\psi=\phi} = S_{\text{p}}(\phi).$$

We include the counter-terms necessary to eliminate the divergences of $S_{\text{p}}(\phi)$; these terms remove the mass and vacuum divergences. So hereafter $S_{\text{p}}(\phi)$ is the renormalized $g\phi^3 - S$ operator. To first order in $\mathscr{L}_{\text{np}}$ we have

$$S_{\text{p}}\left(\psi + \Delta\frac{\delta}{\delta\phi}\right)\mathrm{i}\int \mathrm{d}^4x\mathscr{L}_{\text{np}}(\phi(x))|_{\psi=\phi}.$$

We consider the operator $S_{\text{p}}(\psi + \Delta\delta/\delta\phi)\mathscr{L}_{\text{np}}(\phi)|_{\psi=\phi}$, where $\phi_0 = \phi(x_0)$. We represent this in figure 2, where the ringed dot (the skeleton vertex) denotes $\mathscr{L}_{\text{np}}(\phi(x_0))$, the

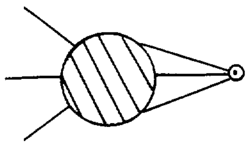

Figure 2. The contribution to the S-operator to first order in the nonpolynomial Lagrangian $\mathscr{L}_{\text{np}}$. The ringed dot $\odot$ denotes the operator $\mathscr{L}_{\text{np}}(\phi_0)$, where $\phi_0 = \phi(x_0)$ and x_0 is the position of the nonpolynomial vertex.

Figure 3. The simplest divergent graphs to first order in $\mathscr{L}_{\text{np}}$. (*a*) is a generalized self-energy, whilst (*b*) is a generalized vacuum, graph. Their explicit contributions are given by equation (9) for (*a*) and (10) for (*b*).

blob is a renormalized ϕ^3-diagram, and the lines to the skeleton vertex each have a factor $\mathrm{d}/\mathrm{d}\phi(x_0) = \xi_0$ as well as the usual propagator Δ. The simplest divergent graphs of this form are shown in figures 3(*a*) and (*b*). We are interested in whether or not such contributions to the S operator give divergences involving the field ϕ at points other than the skeletal vertex x_0. If they don't we say that the divergences are local; otherwise we have nonlocal divergences. In the case of the graph of figure 3(*a*) the divergence is local, which can be seen as follows. We introduce a Pauli–Villars regularization of $\Delta(x)$, so that if M denotes the regularizing mass, the regularized propagator $\Delta(M, x)$ satisfies

$$\Delta(M, x)^2 = A(M)\delta^4(x) + \Delta_f{}^2(M, x)$$

where $\Delta_f{}^2(M, x)$ is a respectable distribution in x and remains so as $M \to \infty$, whilst $A(M)$ behaves as log M for large M. Dropping the M for brevity, we have that the operator contribution of figure 3(*a*) is, with $\xi_0 = \mathrm{d}/\mathrm{d}\phi_0$,

$$3ig \int \mathrm{d}x\, \phi(x)\, \Delta(x - x_0)^2 \xi_0{}^2 \mathscr{L}_{\mathrm{np}}(\phi_0)$$
$$= 3igA\phi_0\xi_0{}^2\mathscr{L}_{\mathrm{np}}(\phi_0) + 3ig \int \mathrm{d}x\, \phi(x)\, \Delta_f{}^2(x - x_0)\xi_0{}^2\mathscr{L}_{\mathrm{np}}(\phi_0).$$

Evidently the divergence in this, the first term on the right of (9), is local. For the contribution described by figure 3(*b*) we have

$$ig \int \mathrm{d}x\, \Delta(x - x_0)^3 \xi_0{}^3 \mathscr{L}_{\mathrm{np}}(\phi_0) = ig\left(\int \mathrm{d}x\, \Delta(x)^3\right)\xi_0{}^3\mathscr{L}_{\mathrm{np}}(\phi_0).$$

The next order in g includes contributions described by figures 4(*a*) and (*b*). Both

(*a*) (*b*)

Figure 4. The simplest nonlocal divergences to first order in $\mathscr{L}_{\mathrm{np}}$, with values given by equation (10) for figure (*a*) and (11) for figure (*b*).

of these give nonlocal divergences, that from figure 4(*a*) being

$$-9g^2 A\phi_0 \int \mathrm{d}x\, \phi(x)\, \Delta_f{}^2(x - x_0)\xi^4 \mathscr{L}_{\mathrm{np}}(\phi_0) \tag{10}$$

and that from figure 4(*b*) being

$$-18g^2 A \int \mathrm{d}x\, \phi(x)\, \Delta_f{}^2(x - x_0)\xi^3 \mathscr{L}_{\mathrm{np}}(\phi_0). \tag{11}$$

Higher order terms in powers of g will also have nonlocal divergences, as will those arising from higher orders in $\mathscr{L}_{\mathrm{np}}$ (where here nonlocality is with respect to the whole set of skeletal vertices). There are also new types of divergences entering in higher orders in $\mathscr{L}_{\mathrm{np}}$; one of these is shown in figure 5.

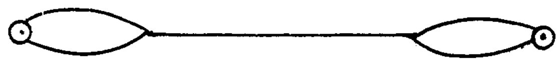

Figure 5. One of the divergences entering to second order in $\mathscr{L}_{\mathrm{np}}$.

The degree of the new divergences is logarithmic when there is one free line carrying an operator ϕ, whilst it is at most quadratic when there are no free lines in the polynomial part of the diagrams. We may thus regard the logarithmic divergences as contributing to an effective 'self-energy' contribution, the quadratic divergences to an effective vacuum amplitude. Let us now turn to see how these divergences may be removed.

4. Counter-terms

We wish to introduce counter-terms which remove the divergences, though we restrict these terms to be local ones. In other words we have to ensure that the nonlocal divergences generated by the total Lagrangian cancel amongst each other. To cancel the lowest order divergences in figures 3(a) and (b) we only need to add the counter-terms

$$-3gA\phi_0\xi_0{}^2\mathscr{L}_{\mathrm{np}}(\phi_0)-gB\xi_0{}^3\mathscr{L}_{\mathrm{np}}(\phi_0) \tag{12}$$

where $B = \int \mathrm{d}x(\Delta(x))^3$. We have now to show that these counter-terms remove the nonlocal divergences (10) and (11); indeed both of these nonlocal terms are removed by taking one power of g from $S_{\mathrm{p}}(\psi+\Delta\delta/\delta\phi)$ together with the first counter term in (12) to give

$$9g^2A\phi_0\int \mathrm{d}x\,\phi(x)\,\Delta_{\mathrm{f}}{}^2(x-x_0)\xi_0{}^4\mathscr{L}_{\mathrm{np}}(\phi_0)+18g^2A\int \mathrm{d}x\,\phi(x)\,\Delta_{\mathrm{f}}{}^2(x-x_0)\xi_0{}^3\mathscr{L}_{\mathrm{np}}(\phi_0).$$

This may be repeated in higher orders in g, provided suitable local counter terms are introduced. In order for this to go through simply in higher orders, it is necessary that the counter terms have the form of functions of ϕ_0 and ξ_0 acting on $\mathscr{L}_{\mathrm{np}}(\phi_0)$. In order to make this cancellation more precise we may proceed as follows.

We begin by classifying all the logarithmic divergences attached to a single skeletal vertex and with one external line. Of these we only consider those which are obtained by constant reinsertions of the second order self-energy graph which has the skeletal vertex as one of its vertices. We consider these graphs because the process of taking the divergent part of the second order self-energy graph causes all these graphs to collapse down to the skeletal vertex; they all give local divergences which have to be taken account of by separate counter terms. We also have to show that the nonlocal divergences generated when subdivergences of such graphs are considered are correctly cancelled by these new counter terms. We denote the sum of all such graphs as $G(x)$ (where we drop the dependence of G on the skeletal vertex x_0 and on ξ_0). G is denoted graphically in figure 6, and includes as a typical graph that shown in figure 7.

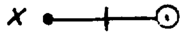

Figure 6. Graphical notation for the total contribution $G(x)$ from all generalized self-energy graphs associated with a given vertex.

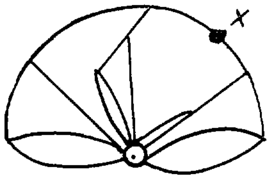

Figure 7. A typical generalized self-energy graph giving a contribution to $G(x)$.

We may sum up all these graphical contributions to obtain a nonlinear integral equation for $G(x)$:

$$G(x) = 3ig\xi_0^2\,\Delta^2(x-x_0)+6ig\xi_0\int dy\,\Delta(x-x_0)\,\Delta(x-y)G(y)$$
$$+3ig\int dy_1\,dy_2\,\Delta(x-y_1)\,\Delta(x-y_2)G(y_1)G(y_2). \qquad (13)$$

We show in Appendix 2 that the solution to this equation is

$$G(x) = B_0\delta^4(x-x_0)+\sum_{n\geq 2} F_n(x)\{ig(\xi_0+B_0)\}^n\mathscr{L}_{\mathrm{np}}(\phi_0) \qquad (14)$$

where $B_0 = \xi_0\Sigma_{n\geq 2}(ig\xi_0)^nA_n$, A_n is the coefficient of $\delta(x-x_0)$ in the locally divergent part of the sum of the nth order graphs, and $F_n(x)$ is the finite part of the sum of the nth order graphs. At this point we might introduce the counter terms $-\phi_0B_0\mathscr{L}_{\mathrm{np}}(\phi_0)$ to eliminate the local divergences, but these would not eliminate the nonlocal divergences which were described earlier.

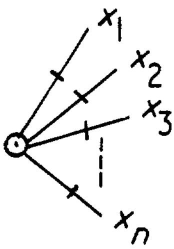

Figure 8. One of the enlarged set of graphs which have to be considered in order that there is cancellation of nonlocal divergences associated with a given skeletal vertex.

To achieve such a cancellation we enlarge the class of graphs by including $\Pi_{i=1}^nG(x_i)$, as shown in figure 8. These contributions may be summed over n to give the operator

$$\exp(\psi_0B_0)\,\exp(\psi F(\xi_0+B_0))\mathscr{L}_{\mathrm{np}}(\phi_0)|_{\psi=\phi} \qquad (15)$$

where

$$\psi F(\xi) = \sum_{n\geq 2}\int dx\,\psi(x)F_n(x)(3ig\xi)^n\,.$$

Introducing the normal-order symbol: : by

$$:\exp(\phi_0B_0): = \sum_{n\geq 0}\frac{1}{n!}\phi_0{}^nB_0{}^n$$

we may write (15) as

$$:\exp(\phi_0 B_0):\exp(\psi F(\xi_0+B_0))\mathscr{L}_{\mathrm{np}}(\phi_0)|_{\psi=\phi}.$$

The identity

$$:\exp(\phi_0 B_0):(\xi_0+B_0) = \xi_0:\exp(\phi_0 B_0):$$

shows that the total contribution from the graphs of figure 8 is then

$$\exp(\psi F(\xi_0)):\exp(\phi_0 B_0):\mathscr{L}_{\mathrm{np}}(\phi_0)|_{\psi=\phi}. \tag{16}$$

We may now introduce the counter terms by replacing $\mathscr{L}_{\mathrm{np}}(\phi)$ by

$$I(\phi,\xi)\mathscr{L}_{\mathrm{np}}(\phi) \tag{17}$$

where

$$I(\phi,\xi) = \{:\exp(\phi B):\}^{-1}$$
$$= 1-\phi B-\tfrac{1}{2}\phi^2 B^2+(\phi B)^2+\dots.$$

Then the contribution (16) reduces to

$$\exp(\psi F(\xi_0))\mathscr{L}_{\mathrm{np}}(\phi_0)|_{\psi=\phi}$$

so that both local and nonlocal divergences are cancelled by local counter-terms.

The second group of divergences are among those graphs without external lines except those coming from the skeletal vertex. We can lump these together into an infinite series in g with infinite coefficients, which we denote by $C(\xi_0)$. These arise from the graphs like those of figure 9 on summation over n. If we sum the graphs of

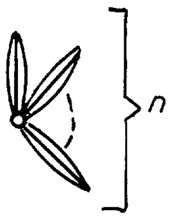

Figure 9. One of the enlarged set of generalized vacuum graphs associated with a skeletal vertex, and whose sum has contribution $C(\xi)$.

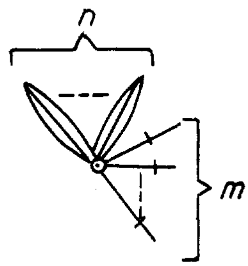

Figure 10. Further enlarged set of graphs with n vacuum graphs and m self-energy graphs associated with a given skeletal vertex.

figure 10 over n and m we obtain the contribution

$$\exp(\psi F(\xi_0)):\exp(\phi_0 B_0):C(\xi_0)\mathscr{L}_{\mathrm{np}}(\phi_0)|_{\psi=\phi}$$

so that the previous total nonpolynomial Lagrangian (17) must be modified tc

$$C(\xi_0)^{-1} I(\phi_0, \xi_0)\mathscr{L}_{\mathrm{np}}(\phi_0). \tag{18}$$

This now includes all the counter terms that are necessary. Hereafter we ignore the parts that give $C(\xi_0)$, for simplicity (they may be easily included). We consider how the total Lagrangian (18) removes all divergences.

We consider first the removal of all divergences to first order in $\mathscr{L}_{\mathrm{np}}$. We draw any graph with ϕ^3 vertices, none of whose ϕ^3 vertices has more than one line joined to the skeletal vertex x_0 (so including, among others, all those considered so far). Then the effect of incorporating the excluded graphs is to dress each of the lines joined to the skeletal vertex by adding to each propagator $\Delta(x-x_0)$ the term $\int\Delta(x-y)G(y)\,\mathrm{d}y$. Then if the original graphical contribution was

$$\int \mathrm{d}\overset{N}{X} G(\phi, \overset{N}{X}) \prod_{i=1}^{N} \Delta(x_i - x_0)\, \xi_0{}^N \mathscr{L}_{\mathrm{np}}(\phi_0)$$

where the ϕ dependence of $G(\phi, \overset{N}{X})$ allows external legs to be taken account of, the corrected contribution, also taking account of the counter terms in (18), is

$$\int \mathrm{d}\overset{N}{X}\, G(\psi, \overset{N}{X}) \prod_{i=1}^{N} \{\Delta(x_i - x_0)(\xi_0 + B_0) + \Delta F_i(\xi_0 + B_0)\}$$

$$\times \exp(\psi_0 B_0) \exp(\psi F(\xi_0 + B_0)) I(\phi_0, \xi_0)\mathscr{L}_{\mathrm{np}}(\phi_0)|_{\psi=\phi} \tag{19}$$

where

$$\Delta F_i(\xi) = \sum_{n \geqslant 2} \int \mathrm{d}x\, \Delta(x_i - x) F_n(x - x_0)(3\mathrm{i}\xi g)^n.$$

If we place $\exp(\psi_0 B)$ to the left we may change it to: $\exp(\phi_0 B_0)$: and then move it to the right, changing $(\xi_0 + B_0)$ to ξ_0 till we reach $I(\phi_0, \xi_0)$, with which it cancels, leaving the finite contribution

$$\int \mathrm{d}\overset{N}{X} G(\psi, \overset{N}{X}) \prod_{i=1}^{N} \{\Delta)x_i - x_0)\xi_0 + \Delta F_i(\xi_0)\} \exp(\psi F(\xi_0))\mathscr{L}_{\mathrm{np}}(\phi_0)|_{\psi=\phi}.$$

To second order in $\mathscr{L}_{\mathrm{np}}$, we show that the same counter terms are sufficient to remove divergences. We have the divergences generated as before on each skeletal vertex separately, to which we must add those arising from lines joining the two

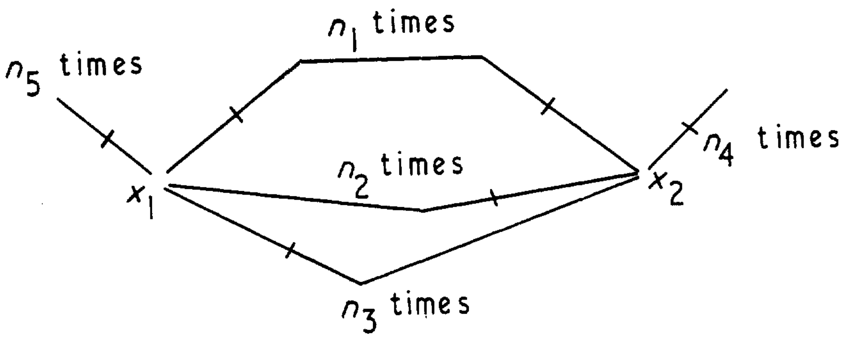

Figure 11. The general type of graph involving divergences associated with one or other or both of two skeletal vertices; the sum of these contributions over $n_1, \ldots, n_5$ is given by equation (20).

skeletal vertices. Thus we have to sum the graphs of figure 11 over the integers $n_1, \ldots, n_5$; with suitable combinatorial coefficients. The result of this may be shown to be

$$\exp\left\{\int \mathrm{d}x\, \psi(x)(G_1(x)+G_2(x))\right\}+\int \mathrm{d}x\, \Delta(x_1-x)\xi_1 G_2(x)+\int \mathrm{d}x\, \Delta(x_2-x)\xi_2 G_1(x)$$
$$+\int \mathrm{d}x\, \mathrm{d}y\, G_1(x)G_2(x)\Delta(x-y)\Big\}\exp\left(\tfrac{1}{2}\frac{\delta}{\delta\phi}\Delta\frac{\delta}{\delta\phi}\right)\prod_{i=1,2} I(\phi_i,\xi_i)\,\mathscr{L}_{\mathrm{np}}(\phi_i)|_{\psi=\phi} \qquad (20)$$

where $G_i(x)$ is the value of $G(x)$ with respect to the ith skeletal vertex, and these are taken to be at x_1 and x_2. We may rewrite (20) as

$$\exp\Big(\psi_1 B_1+\psi_2 B_2+\psi F_1(\xi_1+B_1)+\psi F_2(\xi_2+B_2)+\Delta_1\xi_1 B_2+\Delta_2\xi_2 B_1$$
$$+\xi_2[\Delta F(\xi_1+B_1)]_{12}+\xi_1\{\Delta F(\xi_2+B_2)\}_{12}+\Delta_{12}B_1B_2+B_2\{\Delta F(\xi_1+B_1)\}_{12}$$
$$+B_1\{\Delta F(\xi_2+B_2)\}_{12}+[F(\xi_1+B_1)\Delta F(\xi_2+B_2)]_{12}\Big)\exp\left(\tfrac{1}{2}\frac{\delta}{\delta\phi}\Delta\frac{\delta}{\delta\phi}\right)$$
$$\times\prod_{i=1,2} I(\phi_i,\xi_i)\,\mathscr{L}_{\mathrm{np}}(\phi_i)|_{\psi=\phi} \qquad (21)$$

where

$$[FG]_{12}=\int F(x_1-y)G(y-x_2)\,\mathrm{d}y.$$

If we replace $\exp(\psi_i B_i)$ by $:\exp(\phi_i B_i):$ at the front of (21) we may pull it through, using the same technique as for the single vertex, to obtain

$$\exp\{\psi F_1(\xi_1)+\psi F_2(\xi_2)+\xi_1[\Delta F(\xi_2)]_{12}+\xi_2[\Delta F(\xi_1)]_{12}+[F(\xi_1)\Delta F(\xi_2)]_{12}+\Delta_{12}\xi_1\xi_2\}$$
$$\times\, :\exp(\phi_i B_i):\exp(-\Delta_{12}\xi_1\xi_2)\,\exp\left(\tfrac{1}{2}\frac{\delta}{\delta\phi}\Delta\frac{\delta}{\delta\phi}\right)\prod_{i=1,2} I(\phi_i,\xi_i)\,\mathscr{L}_{\mathrm{np}}(\phi_i)|_{\psi=\phi}. \qquad (22)$$

But $\exp(-\Delta_{12}\xi_1\xi_2)\exp\{\frac{1}{2}(\delta/\delta\phi)\Delta\delta/\delta\phi\}=1$ when operating on functions of ϕ_1 and ϕ_2 only, and then we may use the cancellation process described on the single vertex, to obtain finally the convergent expression

$$\exp\{\psi F_1(\xi_1)+\psi F_2(\xi_2)+\xi_1[\Delta F(\xi_2)]_{12}+\xi_2[\Delta F(\xi_1)]_{12}+[F(\xi_1)\Delta F(\xi_2)]_{12}\}$$
$$\times\,\mathrm{Sk}(x_1,x_2)|_{\psi=\phi}. \qquad (23)$$

We may use exactly the same technique to show how the contribution from any graph may also be cancelled to second order in $\mathscr{L}_{\mathrm{np}}$; since this proceeds almost identically to the discussion given for first order in $\mathscr{L}_{\mathrm{np}}$ we will not consider it further here.

We may also extend the above discussion to the case of general order in $\mathscr{L}_{\mathrm{np}}$; since it is almost identical to the above case of second order we only quote the result. The total contribution to the Nth order in $\mathscr{L}_{\mathrm{np}}$ is obtained by summing only over ϕ^3-graphs which have no vertices with more than one line joined to the skeletal vertices at $x_1, \ldots, x_N$. If such a graph gives a contribution

$$\int \mathrm{d}\overset{M}{Y} G(\psi,\overset{M}{Y})\prod_{i,j}\Delta(y_i-x_j)\xi_j\,\mathrm{Sk}(\overset{N}{X})|_{\psi=\phi}$$

then its dressed contribution, with counter-terms removing the divergences, will be the finite term

$$\int \mathrm{d}\overset{M}{Y}\, G(\psi, \overset{M}{Y}) \prod_{i,j} \{\Delta(y_i - x_j)\xi_j + \Delta F_i(\xi_j)\} \exp(\psi F(\xi_j))\, \mathrm{Sk}(\overset{N}{X})|_{\psi=\phi}. \tag{24}$$

The total contribution to all orders in g and Nth order in $\mathscr{L}_{\mathrm{np}}$ is then obtained by summing the contribution (24) over such reduced graphs G.

One important remark is that it is necessary to show that the terms involving $\xi_1[\Delta F(\xi_2)]_{12}$ or $[F(\xi_1)\Delta F(\xi_2)]_{12}$ in the exponential in (23) produce no further divergences when they are expanded; apparently they do, as the function $F(\xi_i)$ behaves as a δ function in its space–time variables. However, each of these terms involves Δ_{12} always multiplied by $\xi_1\xi_2$. But

$$\begin{aligned} \xi_1\xi_2\, \mathrm{Sk}(1, 2) &= \int_0^\infty \mathrm{d}t_1 \int_0^\infty \mathrm{d}t_2 t_1 t_2 g(t_1) g(t_2) \exp(t_1\phi_1 + t_2\phi_2 + t_1 t_2 \Delta_{12}) \\ &= \left(\frac{\partial}{\partial \Delta_{12}}\right) \mathrm{Sk}(1, 2) \end{aligned}$$

where $\mathscr{L}_{\mathrm{n}\phi}(\phi) = \int_0^\infty \mathrm{d}t\, g(t) \exp(t\phi)$. Thus these potentially dangerous terms always behave like $(\Delta_{12}\partial/\partial\Delta_{12})$ acting on the skeleton, so cannot produce new divergences.

5. Discussion

We would like to stress at this point that the enumeration of divergences, and their removal, described in the previous section is independent of the detailed form of $\mathscr{L}_{\mathrm{np}}$; the nonpolynomial Lagrangian, including the counter terms is

$$C\left(\frac{\mathrm{d}}{\mathrm{d}\phi}\right)^{-1} I\left(\phi, \frac{\mathrm{d}}{\mathrm{d}\phi}\right) \mathscr{L}_{\mathrm{np}}(\phi)$$

where I and C are independent of $\mathscr{L}_{\mathrm{np}}$, and in the expressions for the finite parts of order N in g we have functions of $\mathrm{d}/\mathrm{d}\phi_i$, $i = 1, \ldots, N$, independent of $\mathscr{L}_{\mathrm{np}}$, acting on the skeleton operator $\mathrm{Sk}(\overset{N}{X})$ appropriate to $\mathscr{L}_{\mathrm{np}}(\phi)$. In order that the final results be convergent it is necessary that the skeletons are suitably well defined, and possess no ultraviolet divergences. This was shown to be the case for a certain class of rational functions of the fields, these being of the form

$$\mathscr{L}_{\mathrm{np}}(\phi) = \sum \frac{A_n}{(1+\lambda_n\phi)}$$

with the λ_n all real.

We also have to accept that there are an infinite number of arbitrary constants, due to the fact that a new divergence is introduced in each order of g in $G(x)$. It is useful to note that the infinite number of counter-terms we have found necessary is far fewer than would be required if the simpler method of taking 'regular' perturbation theory in all coupling constants and adding counter terms where needed were used. This is trivial to show, since the 'regular' perturbation theory graphs have divergences of arbitrarily high order, as is seen in the self-energy graph of order $g^2\lambda^{2n+2}$ arising from the interaction $g(1+\lambda\phi)^{-1}$; this has degree of divergence $(2n-4)$. It is through our careful, if somewhat 'laborious', discussion that we have been able to reduce the number of these counter terms, and so the number of arbitrary constants. In fact our manipulations are transparent enough to see that we are not able to make any further reduction in their number. It is possible that these are related when a symmetry, such as chiral symmetry, is present.

It is difficult to extend the above result to higher-degree polynomials. In any case it would not be expected to be possible to introduce a finite set of suitable counter terms if the polynomial Lagrangian had degree greater than four. Even if infinite numbers of counter terms for the polynomial part of the Lagrangian were acceptable there would still be two crucial difficulties. Firstly one may not be able to isolate divergences so that they lie in the neighbourhood of a single skeletal vertex, and so can be absorbed by a counter term localized at that vertex. This is essential for our method of work, otherwise it is possible that nonlocal counter terms are involved. The other problem is that even if there is a localization of divergences it may still be very difficult to disentangle them; in other words there may be very difficult overlap problems. A typical localized divergence for a ϕ^4 polynomial Lagrangian is shown in figure 12; it seems very difficult to specify how to remove the subdivergences in an

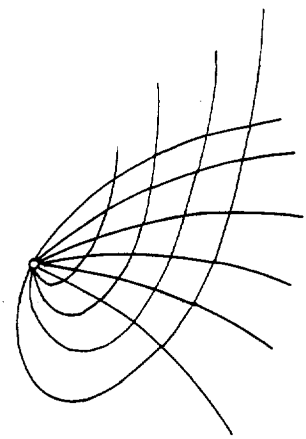

Figure 12. A typical divergent graph associated with a given skeletal vertex when the polynomial part of the interaction Lagrangian is quartic; every vertex of this graph can be collapsed down to the skeletal vertex with a resulting divergent constant. The disentangling of such a reduction process seems very difficult.

unambiguous fashion, as was achieved in the last section for the ϕ^3 case. We remark that we restrict ourselves here to local counter-terms, that is, those constructed from the fields at a single point, since otherwise we would not expect the extension of the general properties of quantum field theories to nonlocalizable theories (Steinmann unpublished), described briefly in the introductory section, to be applicable.

The final question is whether or not the techniques developed here are applicable to the chiral case, where derivatives are present. This does not appear to be immediately apparent, since the divergences are caused in this case by coalescing of skeletal vertices. We hope to return to a fuller discussion of this problem elsewhere.

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Appendix 1

We prove

$$\exp\left(\tfrac{1}{2}\frac{\delta}{\delta\phi}\Delta\frac{\delta}{\delta\phi}\right)F(\phi)G(\phi) = \exp\left(\tfrac{1}{2}\frac{\delta}{\delta\psi}\Delta\frac{\delta}{\delta\psi}\right)F\left(\psi+\Delta\frac{\delta}{\delta\psi}\right)\exp\left(\tfrac{1}{2}\frac{\delta}{\delta\phi}\Delta\frac{\delta}{\delta\phi}\right)G(\phi)|_{\psi=\phi}.$$

as follows.

$$\begin{aligned}
\text{l.h.s.} &= \exp\left(\tfrac{1}{2}\frac{\delta}{\delta\phi}\Delta\frac{\delta}{\delta\phi}\right)\exp(j\phi)F(\phi)G(\phi)|_{j=0} \\
&= F\left(\frac{\delta}{\delta j}\right)\exp(j\phi)\exp\left\{\tfrac{1}{2}\left(\frac{\delta}{\delta\phi}+j\right)\Delta\left(\frac{\delta}{\delta\phi}+j\right)\right\}G(\phi)|_{j=0} \\
&= F\left(\frac{\delta}{\delta j}\right)\exp(j\phi)+\tfrac{1}{2}j\Delta j+j\Delta\frac{\delta}{\delta\psi}\exp\left(\tfrac{1}{2}\frac{\delta}{\delta\phi}\Delta\frac{\delta}{\delta\phi}\right)G(\phi)|_{\psi=\phi,j=0} \\
&= \text{r.h.s.}
\end{aligned}$$

Appendix 2

We expand

$$G(x) = \xi\sum_{n\geqslant 1}(3\mathrm{i}g\xi)^n G_n(x) \tag{A.1}$$

where $G_n(x)$ is independent of ξ, and we take the skeletal vertex to be $x = 0$. The substitution of this in the integral equation (13) gives

$$\begin{aligned}
G_n(x) &= G_1(x)\delta_{n1}+2\int \mathrm{d}y\,\Delta(x)\Delta(x-y)G_{n-1}(y) \\
&\quad+\int \mathrm{d}y\int \mathrm{d}z\,\Delta(x-y)\Delta(x-z)\sum_{r=1}^{n-2}G_r(y)G_{n-1-r}(z).
\end{aligned} \tag{A.2}$$

As above

$$\begin{aligned}
G_1(x) &= \Delta(x)^2 = A\delta(x)+\Delta_{\mathrm{f}}^2(x) \\
G_2(x) &= 2A\Delta(x)^2+2\int \mathrm{d}y\,\Delta(x)\Delta(x-y)\Delta_{\mathrm{f}}^2(y) \\
&= (2A^2+A^1)\delta(x)+2A\,\Delta_{\mathrm{f}}^2(x)+2\left[\int \mathrm{d}y\,\Delta(x)\Delta(x-y)\Delta_{\mathrm{f}}^2(y)\right]_{\mathrm{f}}.
\end{aligned}$$

We conjecture that the generalization of this is

$$G_n(x) = A_n{}^0\delta(x)+\sum_{r=1}^{n-1}A_n{}^rF_r(x)+F_n(x). \tag{A.3}$$

Since F_r contains more than one graph if $r > 2$, the question is whether these graphs occur with the same numerical factors, within an overall constant, in G_n. Substituted in (A.2), this gives, after some algebra, firstly equations for $A_n{}^0, \ldots, A_n{}^{n-1}$

and F_n in terms of $A_m{}^0, \dots, A_m{}^{m-1}$ and F_m, $m < n$:

$$\left.\begin{aligned} A_n{}^0 = 2A_{n-1}{}^0A_1{}^0 + 2\sum_{r=1}^{n-1} A_{n-1}{}^rC_r + \sum_{\substack{t,u\geqslant 1\\ r+s=n-1}} A_r{}^tA_s{}^uD_{t,u} \\ +2\sum_{\substack{t\geqslant 1,\\ r+s=n-1}} A_r{}^tA_s{}^0C_t + A_1{}^0 \sum_{r+s=n-1} A_r{}^0A_s{}^0 \end{aligned}\right\} \tag{A.4}$$

$$A_n{}^1 = 2A_{n-1}{}^0 + \sum_{r+s=n-1} A_r{}^0A_s{}^0 \tag{A.5}$$

$$A_n{}^r = A_{n-1}{}^{r-1} + \sum_{s+t=n-1} A_s{}^{r-1}A_t^0 \qquad (2 \leqslant r \leqslant n-2) \tag{A.6}$$

$$F_r = 2K_{n-1} + \sum_{r+s=n-1} L_{r,s} \tag{A.7}$$

where

$$\int \mathrm{d}y\, \Delta(x)\Delta(x-y)F_n(y) = C_n\delta(x)K_n(x)$$

and

$$\int \mathrm{d}y \int \mathrm{d}z\, \Delta(x-y)\Delta(x-z)F_m(y)F_n(z) = D_{m,n}\delta(x) + L_{m,n}(x)$$

K_n and $L_{m,n}$ being finite. Secondly we obtain the relations

$$A_n{}^{r+s} - A_{n-s}{}^r - A_{n-r}{}^s - \sum_{t,r<t<n-s} A_t{}^rA_{n-t}{}^s + \sum_{t,t\geqslant r+s} A_{n-t}{}^0A_t{}^{r+s} = 0. \tag{A.8}$$

If follows from (A.5) and (A.6) that

$$A_n{}^r = \sum_{\substack{m_i\geqslant 0\\ \sum_i m_i=n-s}} A_m{}^0, \dots, A_{mr+1}{}^0, A_0{}^0 = 1. \tag{A.9}$$

Since (A.8) follows from this, we know that our conjecture was correct. We work in terms of $A_n{}^0$ and F_n, which are independent, since new graphs and divergences arise in each order of g, and make no further use of (A.4) and (A.7). Combining (A.1), (A.2), (A.3) and (A.9) we obtain with $A_n{}^0 = B_n$.

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